## THE UNIVERSITY OF AKRON <br> Mathematics and Computer Science

Lesson 6: Dividing \& Factoring Polynomials

## Directory

- Table of Contents
- Begin Lesson 6


Lesson 6: Dividing \& Factoring Polynomials Table of Contents
6. Dividing \& Factoring Polynomials
6.1. Polynomials: A Quick Review
6.2. Polynomial Division

- Polynomial Division Explained
6.3. Factoring Polynomials: Motivation
6.4. Factoring Polynomials: Theory
- Reducible versus Irreducible Polynomials - The Fundamental Theorem of Algebra • Roots and Linear Factors Related
6.5. Factoring Polynomials: Methods
- Factoring $x^{2}+b x+c$ • Factoring $x^{2}-a^{2} \bullet$ Factoring $a x^{2}+b x+c \bullet$ Factoring $x^{3} \pm a^{3}$


## 6. Dividing \& Factoring Polynomials

### 6.1. Polynomials: A Quick Review

A polynomial in $x$ is an algebraic expression that can be built up through any (finite) combination of additions, subtractions, and multiplications of the symbol $x$ with itself and with numerical constants. The degree of the polynomial is the value of the highest exponent.

Illustration 1. Examples of Polynomials in $x$.
(a) $2 x+1$ has degree 1 . A degree one polynomial is sometimes called linear because its graph is a straight line.
(b) $5 x^{2}-4 x+3$ has degree 2 . A degree two polynomial is called a quadratic polynomial.
(c) $7 x^{3}-4 x^{2}-\frac{2}{3} x+\frac{8}{9}$ has degree 3 . This is called a cubic polynomial.
(d) $x^{4}+x$ has degree 4 .
(e) $x^{45}-5 x^{34}+x^{3}+1$ has degree 45 .
(f) $-3 y^{3}+2 y-y+\frac{1}{2}$ is a polynomial in $y$ of degree 3 .
(g) As a convenience, a constant is considered to be a polynomial of degree 0 . Thus, the algebraic expression 3 may be interpreted as a polynomial of degree 0 .

Sometimes we symbolically denote a polynomial in $x$ by notations such as $P(x)$ or $Q(x)$; polynomials in some other variable such as $y$ would be denoted similarly: $P(y)$ and $Q(y)$. If $P(x)$ is a polynomial in $x$, then we write (type) $\operatorname{deg}(P(x))$ to refer to the degree of $P(x)$.

Polynomials arise naturally in many branches of mathematics and engineering. Polynomials are the basic building blocks used to create numerical approximations such as the ones used by your hand-held calculator. The graphs of certain polynomials have properties that man exploits to create many useful everyday conveniences such as flashlights and satellite dishes.

One of the reasons for the importance of polynomials-and quotients of polynomials-is that their values can be computed by elementary
arithmetic operations: addition, subtraction, multiplication and division. These are the operations a computer is designed to perform quickly and efficiently.

The sum and product of polynomials is again a polynomial. Illustrate this assertion by doing the next exercise. Classify each "answer" as a polynomial and state its degree.

Exercise 6.1. The Algebra of Polynomials. Perform the indicated operations and classify the results.
(a) $\left(4 x^{3}-6 x^{2}+2 x+1\right)+\left(2 x^{3}-3 x+4\right)$
(b) $\left(7 x^{5}-4 x^{3}+12 x-4\right)-\left(5 x^{5}+3 x^{4}+4 x^{3}+2 x-4\right)$
(c) $(3 x-2)\left(x^{2}-2 x+1\right)$
(d) $\left(x^{2}-4\right)\left(x^{2}+4\right)$

Adding and multiplying polynomials were covered by the general methods described in Lesson 3 (addition) and in Lesson 5 (multiplication). In the next paragraph we look at division of polynomials.

Section 6: Dividing \& Factoring Polynomials

### 6.2. Polynomial Division

In this paragraph, we will be primarily interested in dividing polynomials. If $N(x)$ and $D(x)$ are polynomials in $x$ then the expression

$$
\begin{equation*}
\frac{N(x)}{D(x)} \tag{1}
\end{equation*}
$$

is called a rational expression or a rational function.
Illustration 2. Examples of Rational Expressions.
(a) $\frac{2 x+1}{5 x^{2}-4 x+2}$
(b) $\frac{3 x^{3}-2 x^{2}+7 x+1}{x^{2}+3 x-2}$
(c) $\frac{4 x^{12}-7 x^{9}+1}{x^{3}}$
(d) $\frac{4 x^{4}-3 x^{3}+2 x^{2}+8 x-3}{x+3}$

Section 6: Dividing \& Factoring Polynomials

## - Polynomial Division Explained

Now turning to the problem of computing (1), generally, we can only divide a ratio of polynomials when the when the degree of the numerator is greater than or equal to the degree of the denominaotr. In symbols

$$
\text { If } \operatorname{deg}(N) \geq \operatorname{deg}(D), \text { then we can divide } \frac{N(x)}{D(x)}
$$

In this case, when we do divide, then the result looks like this

$$
\begin{equation*}
\frac{N(x)}{D(x)}=Q(x)+\frac{R(x)}{D(x)} \tag{2}
\end{equation*}
$$

where $Q(x)$ and $R(x)$ are polynomials and $\operatorname{deg}(R(x))<\operatorname{deg}(D(x))$. An interesting and important fact to remember is that the representation on the right-hand side of equation (2) is unique. This fact will be exploited in the section on factoring.

Terminology. Given the representation in equation (2), then

- $N(x)$ is called the dividend;
- $D(x)$ is called the divisor;
- $Q(x)$ is called the quotient;
- $R(x)$ is called the remainder.

This is the same terminology used in long division of numbers.
In Illustration 2, the numerator has less degree than the denominator in example (a); consequently, we cannot divide these two polynomials. In examples (b)-(d), the degree of the numerator is equal to or greater than the degree of the denominator; in each of these examples we can further expand the expression by dividing denominator into the numerator.

Division Algorithm. How to divide one polynomial by another and obtain a result of the form (2)
(a) Arrange the terms of $N(x)$ and $D(x)$ so that the powers of the terms are listed in descending order.
(b) Divide the first term of the dividend by the first term of the divisor. This gives the first term of the quotient.

Section 6: Dividing \& Factoring Polynomials
(c) Now, multiply the term of the quotient just computed by the divisor, and subtract this product from the dividend. The result is the remainder.
$\triangleright$ If the degree of the remainder is less than the degree of the divisor, $D(x)$, you are done!
$\triangleright$ If the degree of the remainder is not less than the divisor, $D(x)$, continue by using the remainder just obtained as a new dividend, repeat; i.e., go to (b) to compute the next term of the quotient.

The next example shows how to perform polynomial division. Study this example closely; with a paper and pencil, slowly work through the example-follow my calculations as I work through the Division Algorithm.
Example 6.1. Divide the expression: $\frac{2 x^{3}-3 x^{2}+6 x-4}{x-1}$.

Section 6: Dividing \& Factoring Polynomials

Again, work through these two examples with pencil and paper. Strive to understand how each entry is determined from the Division AlGORITHM.

Example 6.2. Divide denominator into the numerator in parts (b) and (d) in Illustration 2.

Part (c) of Illustration 2 can be handled differently than parts (b) and (d). Why? Read the example to find out why.
Example 6.3. Divide $\frac{4 x^{12}-7 x^{9}+1}{x^{3}}$ that appeared in part (c) of Illustration 2.

Having done a few examples of polynomials, its time for you to try a few. Follow the Division Algorithm and my examples.
Exercise 6.2. Divide: $\frac{2 x^{3}-3 x^{2}+x-1}{x-2}$.
ExErcise 6.3. Divide: $\frac{4 x^{4}+2 x^{2}+x+1}{x^{2}+1}$.

Section 6: Dividing \& Factoring Polynomials
In the next exercise the remainder is zero. Do you recall what that means?
ExErcise 6.4. Divide: $\frac{x^{3}+1}{x+1}$.

### 6.3. Factoring Polynomials: Motivation

Factoring is the reverse process to expanding:

$$
\begin{equation*}
(x+1)(x+2)=x^{2}+3 x+2 . \tag{3}
\end{equation*}
$$

Reading the equation from left-to-right, we are expanding the polynomial $(x+1)(x+2)$; reading the equation from right-to-left we are factoring the polynomial $x^{2}+3 x+2$.
Illustration 3. Factoring has a variety of uses in mathematics. Here is a set of examples to illustrate that assertion.

Section 6: Dividing \& Factoring Polynomials
(a) Simplification of algebraic expressions:

$$
\begin{aligned}
\frac{x^{3}+3 x^{2}+2 x}{x+1} & =\frac{x\left(x^{2}+3 x+2\right)}{x+1} & & \triangleleft \text { factor a common } x \\
& =\frac{x(x+1)(x+2)}{x+1} & & \triangleleft \text { factor again by }(3) \\
& =x(x+2) & & \triangleleft \text { nicely simplified! }
\end{aligned}
$$

Surely you agree, the last expression is preferable to the first.
(b) Solving Equatons: For what values of $x$ is $x^{2}+3 x+2$ equal zero? That is, solve the equation

$$
x^{2}+3 x+2=0
$$

Since $x^{2}+3 x+2=(x+1)(x+2)$ it is now obvious that the only values of $x$ that make $x^{2}+3 x+2$ equal zero are $x=-1$ and $x=-2$.

Section 6: Dividing \& Factoring Polynomials
(c) Simplification for the purpose of numerical calculations. Suppose you wanted to compute, on your calculator, values of the polynomial

$$
x^{4}+4 x^{3}+6 x^{2}+4 x^{+} 1
$$

for $x=3.23344,-27.3234$, and 34.000123 . Even with the aid of your calculator this would be a tedious task, and there is a very good chance that you will make errors. Suppose I told you that the above expression is nothing more than

$$
(x+1)^{4}
$$

Which one would you use to make the calculations?
Summary. Factorization tends to simplify, reduce the number of arithmetical operations performed, and often times yields valuable information about the behavior of the expression.

### 6.4. Factoring Polynomials: Theory

In this section we discuss some important background information and theory: Reducible versus Irreducible Polynomials; the Fundamental Theorem of Algebra; and Roots and Factors Related.

These are important and fundamental concepts you should try to understand.

- Reducible versus Irreducible Polynomials

Some polynomials can be factored, others cannot. Naturally we have a terminology for each situation.

A polynomial that can be factored into a product of polynomials of smaller degree is called a reducible; otherwise, the polynomial is called irreducible.

Illustration 4. Reducible and Irreducible Polynomials.
(a) The polynomial $x^{2}+3 x+2$ is reducible because it can be factored into factors of smaller degree:

$$
x^{2}+3 x+2=(x+1)(x+2)
$$

Section 6: Dividing \& Factoring Polynomials

We have factored a second degree polynomial into a product of linear polynomials.
(b) The polynomial $x^{3}-3 x^{2}+3 x-1$ is reducible because

$$
x^{3}-3 x^{2}+3 x-1=(x-1)^{3} .
$$

Here we say that $(x-1)$ is a linear factor of multiplicity 3 .
(c) The polynomial $x^{3}+x^{2}+x+1$ is reducible because

$$
x^{3}+x^{2}+x+1=(x+1)\left(x^{2}+1\right)
$$

This polynomial factors into linear and quadratic factors.
(d) The polynomials $x^{2}+1$ and $x^{2}-x+1$ are irreducible; that is, it cannot be factored any further. How do I know that? Read on!

Question. It was stated in the last illustration that $x^{2}+1$ is irreducible. Suppose I factor it as follows: $x^{2}+1=\frac{1}{2}\left(2 x^{2}+2\right)$. Since I have factored it into a product of polynomials (the constant $\frac{1}{2}$ may be considered a polynomial of degree 0 ), does this mean that $x^{2}+1$ is
reducible? Review the definitions of reducible and irreducible before you respond.
(a) Yes
(b) No

Question. Consider the factorization: $2 x^{2}+2 x=(2)\left(x^{2}+x\right)$. We have factored a degree two polynomial into a product of a degree 0 polynomial and a degree 2 polynomial. In light of the discussion in the previous Question, does this mean that $2 x^{2}+2 x$ is irreducible?
(a) Yes
(b) No

- The Fundamental Theorem of Algebra

According to the Fundamental Theorem of Algebra, any polynomial of degree greater than zero can be factored into a product of linear and irreducible quadratic factors.

Illustration 5. For example,

$$
\begin{equation*}
x^{4}+2 x^{3}+2 x^{2}+2 x+1=(x+1)^{2}\left(x^{2}+1\right) . \tag{4}
\end{equation*}
$$

Linear Factors: The factor $(x+1)^{2}$ is called a linear factor because $x+1$, the base of the power, is a degree 1 polynomial (degree $1=$ linear). The presence of $(x+1)^{2}$ in the factorization, of course, means that $(x+1)$ appears twice in the factorization

$$
x^{4}+2 x^{3}+2 x^{2}+2 x+1=(x+1)(x+1)\left(x^{2}+1\right) .
$$

but hardly ever write it this way. Here, we say that $(x+1)$ is a linear factor of multiplicity 2, meaning it appears twice in the factorization.

Irreducible Quatdratic Factors: The other factor, $\left(x^{2}+1\right)$, in equation (4) is a second degree polynomial, or a quadratic polynomial. This particular one is irreducible. The exponent of the factor $\left(x^{2}+1\right)=$ $\left(x^{2}+1\right)^{1}$ is 1 meaning this factor has multiplicity 1 ; there is only 1 factor of this type present in the factorization (4).

Thus, the factorization of the polynomial $x^{4}+2 x^{3}+2 x^{2}+2 x+1$ given in equation (4) is the one described in the Fundamental Theorem of Algebra; that is,

$$
x^{4}+2 x^{3}+2 x^{2}+2 x+1=(x+1)^{2}\left(x^{2}+1\right) .
$$

Section 6: Dividing \& Factoring Polynomials
is a factorization into linear and irreducible quadratics factors.
Exercise 6.5. Consider the factorization:

$$
x^{3}(2 x+1)^{4}\left(x^{2}+1\right)^{5}(5-2 x)^{3}\left(x^{2}+x+1\right) .
$$

Classify each factor as linear or irreducible quadratic and state the multiplicity of each factor.

- Roots and Linear Factors Related

There is a relationship between a root of a polynomial and its linear factors. Let's begin by recalling what a root of a polynomial is. Let

$$
P(x)=a x^{3}+b x^{2}+c x+d
$$

be any polynomial. (For illustrative purposes, I've just written a third degree polynomial.) A root of $P(x)$ is any number $r$ such that

$$
\begin{equation*}
P(r)=a r^{3}+b r^{2}+c r+d=0 \tag{5}
\end{equation*}
$$

Section 6: Dividing \& Factoring Polynomials
that is, a root is any number $r$ that causes the polynomial to evaluate to zero. Another way of thinking of a root is, a root is any solution to the equation:

$$
a x^{3}+b x^{2}+c x+d=0 .
$$

Illustration 6. The numbers -1 and -2 are roots of the polynomial $P(x)=x^{2}+3 x+2$ since

$$
\begin{aligned}
& \triangleright P(-1)=(-1)^{2}+3(-1)+2=1-3+2=0 \\
& \triangleright P(-2)=(-2)^{2}+3(-2)+2=4-6+2=0
\end{aligned}
$$

In the next lesson, Lesson 7, we discuss techniques for finding roots of polynomials. There, we will discuss methods of solving equations.

What is the relation between roots and linear factors?
$\triangleright(x-r)$ a factor implies $r$ is a root. Let $P(x)$ be any polynomial and $r$ a number. Clearly, if $(x-r)$ is a factor or $P(x)$, then $r$ is a root of $P(x)$. Indeed, if we assume $(x-r)$ is a factor of $P(x)$ then

$$
P(x)=Q(x)(x-r)
$$

Section 6: Dividing \& Factoring Polynomials
for some polynomial $Q(x)$; that is, $(x-r)$ appears in the factorization of $P(x)$. Now you can see that

$$
P(r)=Q(r)(r-r)=0
$$

This means that $r$ is a root of the polynomial $P(x)$.
$\triangleright r$ a root implies $(x-r)$ a factor. Divide the polynomial $P(x)$ by the polynomial $(x-r)$. The result, by equation (2) will have the form:

$$
\begin{equation*}
\frac{P(x)}{(x-r)}=Q(x)+\frac{R(x)}{(x-r)} \tag{6}
\end{equation*}
$$

where the remainder $R(x)$ has degree less than $x-r$. This means, in this case, that $R(x)$ has degree zero since $x-r$ had degree one. Degree zero polynomials are constants; thus, $R(x)$ is, in fact, a constant. Ill call this constant $R$. We have then

$$
\frac{P(x)}{(x-r)}=Q(x)+\frac{R}{(x-r)} .
$$

Section 6: Dividing \& Factoring Polynomials

Multiply both sides of the equation by $x-r$ to obtain

$$
\begin{equation*}
P(x)=Q(x)(x-r)+R, \quad R=\text { constant } . \tag{7}
\end{equation*}
$$

There is a simple interpretation for the value of $R$. If we replace $x$ by $r$ in equation (7), we see

$$
P(r)=Q(r)(r-r)+R
$$

or,

$$
\begin{equation*}
R=P(r) \tag{9}
\end{equation*}
$$

Now, if $r$ is a root of the polynomial $P(x)$, then from (5), $P(r)=0$. But by (9), the remainder $R=P(r)=0$. Substituting this into equation (7) we get,

$$
P(x)=Q(x)(x-r)+0, \quad \text { or }, \quad P(x)=Q(x)(x-r)
$$

This means that $(x-r)$ is a factor of $P(x)$.
We now summarize the observations of the past few paragraphs.

Section 6: Dividing \& Factoring Polynomials

Summary. Roots and Linear Factors Related:
Let $P(x)$ is a polynomial and $r$ a number. Then $r$ is a root of $P(x)$ if and only if $(x-r)$ is a factor of $P(x)$.

I still haven't explained how I know that $x^{2}+1$ is irreducible. It is clear that for any number $x, x^{2}+1 \neq 0$; therefore, the polynomial $x^{2}+1$ has no roots, hence, it has no linear factors. Now $x^{2}+1$ has degree two, if it is to be factored, it must factor into linear factors. But $x^{2}+1$ has no linear factors. Thus, $x^{2}+1$ is irreducible.

Exercise 6.6. Read carefully the reasoning in the previous paragraph, and apply it to the polynomial $x^{4}+1$. It is clear that for all $x$, $x^{4}+1 \neq 0$ and so $x^{4}+1$ has no roots, hence, has no linear factors. Can we deduce that $x^{4}+1$ is irreducible in the same way as we deduced that $x^{2}+1$ was irreducible?

### 6.5. Factoring Polynomials: Methods

Section 6: Dividing \& Factoring Polynomials

- Factoring $x^{2}+b x+c$

In this section we discuss methods of factoring a second degree polynomial with integer coefficients with leading coefficient of one:

$$
x^{2}+b x+c \quad b, c \in \mathbb{Z}
$$

If the polynomial $x^{2}+b x+c$ is not irreducible, then it can be factored into a product of linear polynomials of the form:

$$
\begin{equation*}
x^{2}+b x+c=\left(x+r_{1}\right)\left(x+r_{2}\right) . \tag{10}
\end{equation*}
$$

The problem is to determine the values of $r_{1}$ and $r_{2}$. This can be done in one of two ways: (1) by trial and error; or (2) the Quadratic Formula. The latter method will be taken up in Lesson 7.

Let's play for a moment. Expand the right-hand side of (10):

$$
\begin{aligned}
x^{2}+b x+c & =\left(x+r_{1}\right)\left(x+r_{2}\right) \\
& =x^{2}+\left(r_{1}+r_{2}\right) x+r_{1} r_{2}
\end{aligned}
$$

Section 6: Dividing \& Factoring Polynomials
Thus, if $x^{2}+b x+c$ can be factored into $\left(x+r_{1}\right)\left(x+r_{2}\right)$, then

$$
\begin{gathered}
r_{1}+r_{2}=b \\
r_{1} r_{2}=c
\end{gathered}
$$

Let's now summarize the results so far in the form of a shadow box.

How to factor $x^{2}+b x+c$, where $b, c \in \mathbb{Z}$
We try to find two number $r_{1}$ and $r_{2}$ such that

$$
\begin{aligned}
r_{1}+r_{2} & =b \\
r_{1} r_{2} & =c
\end{aligned}
$$

In this case, $x^{2}+b x+c=\left(x+r_{1}\right)\left(x+r_{2}\right)$.

Strategy. Given $x^{2}+b x+c$, where $b$ and $c$ are integers: First list all pairs of integers $r_{1}$ and $r_{2}$ the product of which is $c$. Then among all those pairs of numbers, $r_{1}$ and $r_{2}$, thus listed, choose the pair whose sum is $b$-choose that pair $r_{1}$ and $r_{2}$ such that $r_{1}+r_{2}=b$.

Section 6: Dividing \& Factoring Polynomials
Let's go to the examples.
Example 6.4. Factor each of the following polynomials.
(a) $x^{2}-x-2$
(b) $x^{2}+5 x+6$
(c) $4-3 x-x^{2}$

Solve the next exercise using good notation. Be neat, be organized. Take pride in your work!

Exercise 6.7. Factor each of the following using the recommended strategy as illustrated in Example 6.4.
(a) $x^{2}+7 x+10$
(b) $x^{2}-7 x+10$
(c) $x^{2}-3 x-10$
(d) $x^{2}+3 x-10$
(e) $x^{2}+11 x+10$
(f) $x^{2}-9 x-10$

Here's a few more. Don't forget the strategy and the standard methods as illustrated in Example 6.4.

Exercise 6.8. Factor each of the following.
(a) $x^{2}+4 x-12$
(b) $x^{2}+3 x-18$
(c) $x^{2}-10 x+21$
(d) $x^{2}+7 x-8$
(e) $x^{2}-2 x+1$
(f) $2 x^{2}+8 x+8$

Section 6: Dividing \& Factoring Polynomials

- Factoring $x^{2}-a^{2}$

Polynomials that are in the form of a difference of two squares are easy to factor. From equation (3) of LESSON 5 we have

$$
\begin{equation*}
x^{2}-a^{2}=(x-a)(x+a) \tag{11}
\end{equation*}
$$

The problem students have is recognizing the presence of such a polynomial. It's a matter of training you eyes and brain to work together.

## Illustration 7. Difference of Two Squares.

(a) $x^{2}-1=(x-1)(x+1)$.
(b) $x^{2}-4=(x-2)(x+2)$.
(c) $x^{2}-3=(x-\sqrt{3})(x+\sqrt{3})$. Note: Any positive number can be thought of as the square of another number. Consequently, the $a^{2}$ that is in equation (11) does not have to be a 'perfect square'; it can be any positive number.

Exercise 6.9. (Skill Level 0) Factor each of the following. (a) $x^{2}-9$ (b) $x^{2}-12$ (c) $x^{2}-17$ (d) $25-x^{2}$

Section 6: Dividing \& Factoring Polynomials

Factorization formula (11) came from expansion formula (3) of LesSON 5 . The expansion formula is actually more general: We can factor any difference of two squares - as illustrated below.

## Illustration 8. Difference of two squares.

(a) $4 x^{2}-9=(2 x-3)(2 x+3)$.
(b) $3 x^{2}-16=(\sqrt{3} x-4)(\sqrt{3} x+4)$.
(c) $x^{4}-16=\left(x^{2}-4\right)\left(x^{2}+4\right)=(x-2)(x+2)\left(x^{2}+4\right)$. Here we have applied equation (11) twice! The result is a complete factorization of $x^{4}-16$ into a product of linear and irreducible quadratic factors as prescribed by the Fundamental Theorem of AlGEBRA.

Exercise 6.10. Factor each of the following using (11).

$$
\text { (a) } 4 x^{2}-9 \text { (b) } 5 x^{2}-3 \text { (c) } x^{4}-25
$$

The factorization formula (11) can be applied whenever we have a difference of two squares.

Exercise 6.11. Factor each of the following differences of squares and simplify when possible. Note: 'simplify' $\neq$ 'expand'.

Section 6: Dividing \& Factoring Polynomials

$$
\text { (a) } x^{2}-(x-1)^{2}\left(\text { b) } x^{2} y^{2}-4 \quad \text { (c) } \frac{(2 x-1)^{2}-(2 x+1)^{2}}{x}\right.
$$

- Factoring $a x^{2}+b x+c$

We now turn to the problem of factoring a quadratic polynomial of the form

$$
\begin{equation*}
a x^{2}+b x+c \quad a, b, c \in \mathbb{Z} \tag{12}
\end{equation*}
$$

The trial and error methods illustrated earlier can be utilized to factor (12). Generally, if the polynomial is not irreducible, its factorization of would look like this:

$$
\begin{equation*}
a x^{2}+b x+c=\left(q_{1} x+r_{1}\right)\left(q_{2} x+r_{2}\right) \tag{13}
\end{equation*}
$$

If we multiply out the right-hand side of (13) that will give us clues how to find the numbers $q_{1}, q_{2}, r_{1}$, and $r_{2}$ :

$$
\begin{aligned}
a x^{2}+b x+c & =\left(q_{1} x+r_{1}\right)\left(q_{2} x+r_{2}\right) \\
& =q_{1} q_{2} x^{2}+\left(q_{1} r_{2}+r_{1} q_{2}\right) x+r_{1} r_{2}
\end{aligned}
$$

Section 6: Dividing \& Factoring Polynomials

Thus, we seek numbers such that

$$
q_{1} q_{2}=a \quad r_{1} r_{2}=c \quad q_{1} r_{2}+q_{2} r_{1}=b
$$

The task of finding the numbers $q_{1}, q_{2}, r_{1}$, and $r_{2}$ may be easy or difficult.

Important Point. It is known that if the polynomial $a x^{2}+b x+c$ has only rational roots - roots that are rational numbers-then $q_{1}, q_{2}, r_{1}$, and $r_{2}$ are integers. However, a quadratic polynomial such as (12) may have irrational roots.

In this section, we look at quadratics having rational roots, hence the factors we seek have integer coefficients. The other case can be handled by use of the binomial formula taken up in LESSON 7.

Strategy. To factor $a x^{2}+b x+c$ into to linear factors

$$
a x^{2}+b x+c=\left(q_{1} x+r_{1}\right)\left(q_{2}+r_{2}\right)
$$

(a) List all integer pairs $q_{1}$ and $q_{2}$ such that $q_{1} q_{2}=a$, the coefficient of the $x^{2}$ term.

Section 6: Dividing \& Factoring Polynomials
(b) List all integer pairs $r_{1}$ and $r_{2}$ such that $r_{1} r_{2}=c$, the constant term.
(c) Find that combination of $q_{1}, q_{2}$ and $r_{1}, r_{2}$ such that $q_{1} r_{2}+q_{2} r_{1}=$ $b$, the coefficient of the cross-product term.

This can usually be carried out by first choosing candidates for $q_{1}$ and $q_{2}$ then trying all combinations for $r_{1}$ and $r_{2}$. If that fails, try another choice for $q_{1}$ and $q_{2}$. (I told you it was 'trial and error.')

Here's an example of the scheme just described.
Example 6.5. Factor $6 x^{2}+x-1$.
In the exercises below, use Example 6.5 as a guide to factoring.
Exercise 6.12. Factor $8 x^{2}+2 x-1$.
Exercise 6.13. Factor $6 x^{2}-5 x-6$.
In Lesson 7 we'll review techniques of solving equations. Solving polynomial equations has applications to factorization. (Recall the paragraph on Roots and Linear Factors Related.) At that time we will

Section 6: Dividing \& Factoring Polynomials
be able to factor quadratics with irrational roots as well as rational roots.

- Factoring $x^{3} \pm a^{3}$

The sum or difference of cubes is easy to factor so we'll finish this lesson by factoring this type. The formula we shall use is

$$
\begin{align*}
& b^{3}-a^{3}=(b-a)\left(b^{2}+a b+a^{2}\right) \\
& b^{3}+a^{3}=(b+a)\left(b^{2}-a b+a^{2}\right) \tag{14}
\end{align*}
$$

The major problem of students is recognizing the presence of cubes.
Illustration 9. Factor each of the following.
(a) $x^{3}-8=(x-2)\left(x^{2}+2 x+4\right)$. Here $a=2$ in (14).
(b) $x^{3}+8=(x+2)\left(x^{2}-2 x+4\right)$.
(c) $8 x^{3}-1=(2 x-1)\left(4 x^{2}+2 x+1\right)$. Here $b=2 x$ and $a=1$.
(d) $x^{3} y^{3}+1=(x y+1)\left(x^{2} y^{2}-x y+1\right)$. Here $b=x y$ and $a=1$.

Exercise 6.14. (Skill Level 0) Factor each of the following.
(a) $x^{3}-1$ (b) $27 x^{3}-8$ (c) $8 y^{6}+27$

Section 6: Dividing \& Factoring Polynomials
Exercise 6.15. (Skill Level 0.5) Factor and simplify each.

$$
\text { (a) } \frac{x^{3}-1}{x-1} \text { (b) } \frac{\left(8 x^{3}-1\right)\left(x^{2}+3 x+2\right)}{x^{2}-1} \text { (c) } \frac{8 x^{3}-27}{4 x^{2}-9}
$$

This is the end of LESSON 6 on dividing polynomials and factoring. To continue, go to Lesson 7.

## Solutions to Exercises

6.1. Solutions:
(a) Combine $\left(4 x^{3}-6 x^{2}+2 x+1\right)+\left(2 x^{3}-3 x+4\right)$.

$$
\left(4 x^{3}-6 x^{2}+2 x+1\right)+\left(2 x^{3}-3 x+4\right)=6 x^{3}-6 x^{2}-x+5
$$

This is a polynomial of degree 3 .
(b) $\left(7 x^{5}-4 x^{3}+12 x-4\right)-\left(5 x^{5}+3 x^{4}+4 x^{3}+2 x-4\right)$

$$
\begin{aligned}
\left(7 x^{5}\right. & \left.-4 x^{3}+12 x-4\right)-\left(5 x^{5}+3 x^{4}+4 x^{3}+2 x-4\right) \\
& =2 x^{5}-3 x^{4}-8 x^{3}+10 x
\end{aligned}
$$

which is a polynomial of degree 5 .

Solutions to Exercises (continued)
(c) $(3 x-2)\left(x^{2}-2 x+1\right)$. Use the General Multiplication Rule to obtain

$$
\begin{aligned}
(3 x- & 2)\left(x^{2}-2 x+1\right) \\
& =(3 x)\left(x^{2}\right)-(3 x)(2 x)+(3 x)-2\left(x^{2}\right)+2(2 x)-2 \\
& =3 x^{3}-6 x^{2}+3 x-2 x^{2}+4 x-2 \\
& =3 x^{3}-8 x^{2}+7 x-2
\end{aligned}
$$

Alternately, you can expand as follows:

$$
\begin{aligned}
& x^{2}-2 x+1 \\
& \frac{3 x-2}{3 x^{3}-6 x^{2}+3 x} \\
& \frac{-2 x^{2}+4 x-2}{3 x^{3}-8 x^{2}+7 x-2} \\
& \Longleftarrow \text { The is a polynomial of degree } 3 \text {. }
\end{aligned}
$$

(d) $\left(x^{2}-4\right)\left(x^{2}+4\right)=x^{4}-16$. by (3) in Lesson 5 . A polynomial of degree 4 .

Solutions to Exercises (continued)
6.2. Solution: Just follow the Division Algorithm!

|  | $2 x^{2}+x+3$ | $\triangleleft$ the quotient |
| :---: | :---: | :---: |
| $x-2$ | $2 x^{3}-3 x^{2}+x-1$ | $\triangleleft(\mathrm{b})$ divide $2 x^{3}$ by $x$ to get $2 x^{2}$ |
|  | $2 x^{3}-4 x^{2}$ | $\triangleleft$ (c) multiply $2 x^{2}$ by $x-2$ and place it here |
|  | $x^{2}+x-1$ | $\triangleleft$ (c) subtract; (b) divide $x^{2}$ by $x$ to get $x$ |
|  | $x^{2}-2 x$ | $\triangleleft$ (c) multiply $x$ by $x-2$ and place it here |
|  | $3 x-1$ | $\triangleleft$ (c) subtract; (b) divide $3 x$ by $x$ to get 3 |
|  | $3 x-6$ | $\triangleleft$ (c) multiply 3 by $x-2$ and place it here |
|  | 5 | $\triangleleft$ (c) subtract; done; this is the remainder |

Interpretation of Calculations: $x-2$ divides into $2 x^{3}-3 x^{2}+x-1$ $\ldots 2 x^{2}+x+3$ times with a remainder of 5 . Thus,

$$
\frac{2 x^{3}-3 x^{2}+x-1}{x-2}=2 x^{2}+x+3+\frac{5}{x-2}
$$

Compare this equation with (2).

Solutions to Exercises (continued)
6.3. Solution: Just follow the Division Algorithm!

$$
\begin{aligned}
& 4 x^{2}-2 \\
x^{2}+1 & \triangleleft \text { the quotient } \\
\hline 4 x^{4}+2 x^{2}+x+1 & \triangleleft(\mathrm{~b}) \text { divide } 4 x^{4} \text { by } x^{2} \text { to get } 4 x^{2} \\
\frac{4 x^{4}+4 x^{2}}{-2 x^{2}+x+1} & \begin{array}{l}
\triangleleft \text { (c) multiply } 4 x^{2} \text { by } x^{2}+1 \text { and place it here } \\
\triangleleft \text { (c) subtract; (b) divide }-2 x^{2} \text { by } x^{2} \text { to get }-2
\end{array} \\
& \frac{-2 x^{2}-2}{x+3}
\end{aligned} \begin{aligned}
& \triangleleft \text { (c) multiply }-2 \text { by } x^{2}+1 \text { and place it here } \\
& \text { (c) subtract; done; this is the remainder }
\end{aligned}
$$

Interpretation of Calculations: $x^{2}+1$ divides into $4 x^{4}+2 x^{2}+x+1$ $\ldots 4 x^{2}-2$ times with a remainder of $x+3$. Thus,

$$
\frac{4 x^{4}+2 x^{2}+x+1}{x^{2}+1}=4 x^{2}-2+\frac{x+3}{x^{2}+1}
$$

Compare this equation with (2).

Solutions to Exercises (continued)
6.4. Solution:

$$
\begin{aligned}
& \begin{array}{l|l} 
& x^{2}-x+1 \\
\cline { 2 - 2 }+1
\end{array} \\
& \triangleleft \text { the quotient } \\
& \triangleleft(\mathrm{b}) \text { divide } x^{3} \text { by } x \text { to get } x^{2} \\
& \frac{x^{3}+x^{2}}{-x^{2}+1} \\
& \frac{-x^{2}-x}{x+1} \\
& \begin{array}{r}
x+1 \\
0
\end{array} \\
& \triangleleft \text { (c) multiply } x^{2} \text { by } x+1 \text { and place it here } \\
& \triangleleft \text { (c) subtract; ( } \mathrm{b} \text { ) divide }-x^{2} \text { by } x \text { to get }-x \\
& \triangleleft \text { (c) multiply }-x \text { by } x+1 \text { and place it here } \\
& \triangleleft \text { (c) subtract; (b) divide } x \text { by } x \text { to get } 1 \\
& \triangleleft \text { (c) multiply } 1 \text { by } x+1 \text { and place it here } \\
& \triangleleft \text { (c) subtract; done; this is the remainder }
\end{aligned}
$$

Interpretation of Calculations: $x+1$ divides into $x^{3}+1 \ldots x^{2}-x+1$ times with a remainder of 0 . Thus,

$$
\frac{x^{3}+1}{x+1}=x^{2}-x+1+\frac{0}{x+1}=x^{2}-x+1
$$

Solutions to Exercises (continued)
This means that $x+1$ evenly divides into $x^{3}+1$. If we multiply both sides of the equations by $x+1$ we get

$$
x^{3}+1=(x+1)\left(x^{2}-x+1\right)
$$

The division process can be used as a technique to factor a polynomial.
Exercise 6.4.

Solutions to Exercises (continued)
6.5. Answers:
Factor
$x^{3}$
$(2 x+1)^{4} \quad$ linear
$\left(x^{2}+1\right)^{5} \quad$ irreducible quadratic
$(5-2 x)^{3} \quad$ linear
$\left(x^{2}+x+1\right) \quad$ irreducible quadratic

Classification
linear
linear
irreducible quadratic
linear
$\left(x^{2}+x+1\right) \quad$ irreducible quadratic

3
Multiplicity
3
4
5

1
6.6. No! $x^{4}+1$ has no linear factors, but, by the Fundamental Theorem of Algebra it can be factored into a product of linear factors and irreducible quadratic factors. As it doesn't have any linear factors, and its degree is 4 , it is must be possible to factor it into a product of two irreducible quadratic factors.

Can you find these two factors?

Solutions to Exercises (continued)
6.7. Answers: Hopefully, you used the method illustrated in ExamPLE 6.4.
(a) $x^{2}+7 x+10=(x+2)(x+5)$.
(b) $x^{2}-7 x+10=(x-2)(x-5)$.
(c) $x^{2}-3 x-10=(x+2)(x-5)$.
(d) $x^{2}+3 x-10=(x-2)(x+5)$.
(e) $x^{2}+11 x+10=(x+1)(x+10)$.
(f) $x^{2}-9 x-10=(x+1)(x-10)$.

Did I list all possible combinations, or did I miss one or two?
Exercise 6.7. $\quad$

Solutions to Exercises (continued)
6.8. Answers:
(a) $x^{2}+4 x-12=(x-2)(x+6)$.
(b) $\left.x^{2}+3 x-18=\right)=(x+6)(x-3)$.
(c) $x^{2}-10 x+21=(x-3)(x-7)$.
(d) $x^{2}+7 x-8=(x-1)(x+8)$.
(e) $x^{2}-2 x+1=(x-1)(x-1)=(x-1)^{2}-$ a perfect square!
(f) $2 x^{2}+8 x+8=2\left(x^{2}+4 x+4\right)=2(x+2)(x+2)=2(x+2)^{2}$-again a perfect $\square$ !

Solutions to Exercises (continued)
6.9. Answers:
(a) $x^{2}-9=(x-3)(x+3)$.
(b) $x^{2}-12=(x-\sqrt{12})(x+\sqrt{12})=(x-2 \sqrt{3})(x+2 \sqrt{3})$. You did simplify, didn't you?
(c) $x^{2}-17=(x-\sqrt{17})(x+\sqrt{17})$.
(d) $25-x^{2}=(5-x)(5+x)=-(x-5)(x+5)$. We usually like the $x$-term to come first.

These factorizations are rather boring.

Solutions to Exercises (continued)
6.10. Answers:
(a) $4 x^{2}-9=(2 x-3)(2 x+3)$.
(b) $5 x^{2}-3=(\sqrt{5} x-\sqrt{3})(\sqrt{5} x+\sqrt{3})$.
(c) $x^{4}-25=\left(x^{2}-5\right)\left(x^{2}+5\right)=(x-\sqrt{5})(x+\sqrt{5})\left(x^{2}+5\right)$.

Exercise 6.10.

Solutions to Exercises (continued)
6.11. Solutions:
(a) $x^{2}-(x-1)^{2}=[x-(x-1)][x+(x-1)]=[1][2 x-1]=2 x-1$
(b) $x^{2} y^{2}-4=(x y-2)(x y+2)$
(c) Factor: $\frac{(2 x-1)^{2}-(2 x+1)^{2}}{x}$.

$$
\begin{aligned}
&\left.\frac{(2 x-}{}-1\right)^{2}-(2 x+1)^{2} \\
& x \\
&=\frac{[(2 x-1)-(2 x+1)][(2 x-1)+(2 x+1)]}{x} \\
&=\frac{[-2][4 x]}{x}=-\frac{8 x}{x} \quad \triangleleft \text { cancel the } x! \\
&=-8
\end{aligned}
$$

That simplified down nicely!

Solutions to Exercises (continued)
6.12. Solution:

$$
8 x^{2}+2 x-1=(4 x-1)(2 x+1) .
$$

Note: You really don't need the answers, just multiply out your factorization. If it expands to $8 x^{2}+2 x-1$, you are right!

Solutions to Exercises (continued)
6.13. Solution:

$$
6 x^{2}-5 x-6=(3 x+2)(2 x-3)
$$

Exercise 6.13.

Solutions to Exercises (continued)
6.14. Answers:
(a) $x^{3}-1=(x-1)\left(x^{2}+x+1\right)$
(b) $27 x^{3}-8=(3 x-2)\left(9 x^{2}+6 x+4\right) .(b=3 x$ and $a=2)$
(c) $8 y^{6}+27=\left(2 y^{2}+3\right)\left(4 y^{4}-6 y^{2}+9\right) \quad\left(b=2 y^{2}\right.$ and $\left.a=3\right)$

Exercise 6.14.

Solutions to Exercises (continued)
6.15. Answers:
(a) $\frac{x^{3}-1}{x-1}=\frac{(x-1)\left(x^{2}+x+1\right)}{x-1}=x^{2}+x+1$
(b) Factor and simplify: $\frac{\left(8 x^{3}-1\right)\left(x^{2}+3 x+2\right)}{x^{2}-1}$.

$$
\begin{aligned}
& \frac{\left(8 x^{3}-1\right)\left(x^{2}+3 x+2\right)}{x^{2}-1} \\
& \quad=\frac{(2 x-1)\left(4 x^{2}+2 x+1\right)(x+1)(x+2)}{(x-1)(x+1)} \\
& \quad=\frac{(2 x-1)\left(4 x^{2}+2 x+1\right)(x+2)}{(x-1)}
\end{aligned}
$$

(c) $\frac{8 x^{3}-27}{4 x^{2}-9}=\frac{2 x-3)\left(4 x^{2}+6 x+9\right)}{(2 x-3)(2 x+3)}=\frac{4 x^{2}+6 x+9}{2 x+3}$

Exercise 6.15.

## Solutions to Examples

6.1. Solution: The terms of the divisor, $x-1$, and the dividend, $2 x^{3}-3 x^{2}+6 x-1$, are already arranged in decreasing order. (Step (a) of the Division Algorithm.

The calculations outlined in the Division Algorithm can be arranged into a convenient table format. The method of division and the table is similar to long division of numbers.

$$
\begin{aligned}
& 2 x^{2}-x+5 \quad \triangleleft \text { the quotient } \\
& x - 1 \longdiv { 2 x ^ { 3 } - 3 x ^ { 2 } + 6 x - 4 } \triangleleft ( \mathrm { b } ) \text { divide } 2 x^{3} \text { by } x \text { to get } 2 x^{2} \\
& 2 x^{3}-2 x^{2} \triangleleft \text { (c) multiply } 2 x^{2} \text { by } x-1 \text { and place it here } \\
& -x^{2}+6 x-4 \quad \triangleleft(\mathrm{c}) \text { subtract; (b) divide }-x^{2} \text { by } x \text { to get }-x \\
& -x^{2}+x \quad \triangleleft \text { (c) multiply }-x \text { by } x-1 \text { and place it here } \\
& 5 x-4 \quad \triangleleft \text { (c) subtract; (b) divide } 5 x \text { by } x \text { to get } 5 \\
& 5 x-5 \triangleleft(\mathrm{c}) \text { multiply } 5 \text { by } x-1 \text { and place it here } \\
& \triangleleft \text { (c) subtract; Done. This is the remainder }
\end{aligned}
$$

Interpretation of Calculations: $x-1$ goes into $2 x^{3}-3 x^{2}+6 x-4 \ldots$ $2 x^{2}-x+5$ times with a remainder of 1 . Thus,

$$
\frac{2 x^{3}-3 x^{2}+6 x-4}{x-1}=2 x^{2}-x+5+\frac{1}{x-1}
$$

Compare this equation with (2). The quotient is $Q(x)=2 x^{2}-x+5$ and the remainder is $R(x)=1$. Note that $\operatorname{deg}(R(x))=0<1$, as advertised above.

The validity of the above equation can be verified by getting a common denominator of the right-hand side of the equation and observing that it results in the left-hand side!

Solutions to Examples (continued)
6.2. Problem (b) Divide: $\frac{3 x^{3}-2 x^{2}+7 x+1}{x^{2}+3 x-2}$.

Solution to (b)

$$
\begin{aligned}
& 3 x-11 \quad \triangleleft \text { the quotient } \\
& x ^ { 2 } + 3 x - 2 \longdiv { 3 x ^ { 3 } - 2 x ^ { 2 } + 7 x + 1 } \quad \triangleleft ( \mathrm { b } ) \text { divide } 3 x^{3} \text { by } x^{2} \text { to get } 3 x \\
& 3 x^{3}+9 x^{2}-6 x \quad \triangleleft \text { (c) multiply } 3 x \text { by } x^{2}+3 x-2 \\
& -11 x^{2}+13 x+1 \quad \triangleleft \text { (c) subtract; then repeat (b) } \\
& -11 x^{2}-33 x+22 \quad \triangleleft(\mathrm{c}) \text { multiply }-11 \text { by } x^{2}+3 x-2 \\
& 46 x-21 \quad \triangleleft \text { (c) subtract; Done. The remainder }
\end{aligned}
$$

Thus,

$$
\frac{3 x^{3}-2 x^{2}+7 x+1}{x^{2}+3 x-2}=3 x-11+\frac{46 x-21}{x^{2}+3 x-2}
$$

Notice that the degree of the remainder is less than the degree of the divisor.

Solutions to Examples (continued)
Solution to (d) Divide: $\frac{4 x^{4}-3 x^{3}+2 x^{2}+8 x-3}{x+3}$

|  | $4 x^{3}-15 x^{2}+47 x-133$ | $\triangleleft$ the quotient |
| :---: | :---: | :---: |
| $x+3$ | $\begin{aligned} & 4 x^{4}-3 x^{3}+2 x^{2}+8 x-3 \\ & 4 x^{4}+12 x^{3} \end{aligned}$ | $\triangleleft$ (b) divide <br> $\triangleleft$ (c) multiply |
|  | $\begin{aligned} & -15 x^{3}+2 x^{2}+8 x-3 \\ & -15 x^{3}-45 x^{2} \end{aligned}$ | $\triangleleft$ (c) subtract; (b) divide <br> $\triangleleft$ (c) multiply |
|  | $47 x^{2}+8 x-3$ | $\triangleleft$ (c) subtract; (b) divide |
|  | $47 x^{2}+141 x$ | $\triangleleft$ (c) multiply |
|  | $-133 x-3$ | $\triangleleft$ (c) subtract; (b) divide |
|  | -133x-399 | $\triangleleft$ (c) multiply |
|  | 396 | $\triangleleft$ (c) subtract and Done. |

Thus,

$$
\frac{4 x^{4}-3 x^{3}+2 x^{2}+8 x-3}{x+3}=4 x^{3}-15 x^{2}+47 x-133+\frac{396}{x+3}
$$

Solutions to Examples (continued)
6.3. Solution: Here the denominator, the divisor, is a monomial: a single term. In this case, we don't have to carry out the elaborate division process. We simply ...

$$
\begin{aligned}
\frac{4 x^{12}-7 x^{9}+1}{x^{3}} & =\frac{4 x^{12}}{x^{3}}-\frac{7 x^{9}}{x^{3}}+\frac{1}{x^{3}} \\
& =4 x^{9}-7 x^{6}+\frac{1}{x^{3}}
\end{aligned}
$$

That is, we divide by separation of fractions. Thus,

$$
\frac{4 x^{12}-7 x^{9}+1}{x^{3}}=4 x^{9}-7 x^{6}+\frac{1}{x^{3}}
$$

When a shorter method works ... use the shorter method!

Solutions to Examples (continued)
6.4. Solutions:
(a) Factor $x^{2}-x-2$. Here we seek two numbers $r_{1}$ and $r_{2}$ such that $r_{1}+r_{2}=-1$ and $r_{1} r_{2}=-1$.
Solution: List all pairs of integers the product of which is -2 .

| $r_{1}$ | $r_{2}$ | $r_{1} r_{2}$ | $r_{1}+r_{2}$ |
| :---: | :---: | :---: | :---: |
| -1 | 2 | 2 | 1 |
| -2 | 1 | 2 | $-1 \Longleftarrow$ This is it |

Therefore, the factorization is for $r_{1}=-2$ and $r_{2}=1$ :

$$
\begin{aligned}
x^{2}-x-2 & =\left(x+r_{1}\right)\left(x+r_{2}\right) \\
& =(x+(-2))(x+1) \\
& =(x-2)(x+1) .
\end{aligned}
$$

Or,

$$
x^{2}-x-2=(x-2)(x+1) \text {. }
$$

(b) Factor $x^{2}+5 x+6$.

Solution: We follow the strategy suggested above. List all pairs of integers the product of which is 6 , and we search for that pair whose sum is 5 .

| $r_{1}$ | $r_{2}$ | $r_{1} r_{2}$ | $r_{1}+r_{2}$ |  |
| ---: | ---: | :---: | :---: | :--- |
| 1 | 6 | 6 | 7 |  |
| -1 | -6 | 6 | -7 |  |
| 2 | 3 | 6 | 5 | $\Longleftarrow$ This is it |

Notice that I did not list out all possible combinations. I stopped as soon as I found the proper pair: $r_{1}=2$ and $r_{2}=3$. The factorization is

$$
x^{2}+5 x+6=(x+2)(x+3)
$$

Solutions to Examples (continued)
(c) Factor $4-3 x-x^{2}$.

Solution: This has a slight twist. The coefficient of the $x^{2}$ term is not one. We just factor out -1 to get

$$
4-3 x-x^{2}=-\left(x^{2}+3 x-4\right)
$$

and factor $x^{2}+3 x-4$. Again, we follow the strategy suggested above. List all pairs of integers the product of which is -4 , and we search for that pair whose sum is 3 .

| $r_{1}$ | $r_{2}$ | $r_{1} r_{2}$ | $r_{1}+r_{2}$ |
| ---: | :---: | :---: | :---: |
| 1 | -4 | -4 | -3 |
| -1 | 4 | -4 | 3 |
|  |  | $=$ Found them! |  |

The factorization is

$$
4-3 x-x^{2}=-\left(x^{2}+3 x-4\right)=-(x-1)(x+4)
$$

Or, we could, perhaps write is as follows:

$$
4-3 x-x^{2}=(1-x)(x+4)
$$

Solutions to Examples (continued)
6.5. Solution: This is not a pretty method. The problem is to factor $6 x^{2}+x-1$.

Trial 1: Guess a factorization of the form $\left(6 x+r_{1}\right)\left(x+r_{1}\right)$. The values of $r_{1}$ and $r_{2}$ are such that $r_{1} r_{2}=-1$.

$$
\begin{aligned}
& \text { Try: } r_{1}=1, r_{2}=-1: \quad(6 x+1)(x-1)=6 x^{2}-5 x-1 \quad \text { Error! } \\
& \text { Try: } r_{1}=-1, r_{2}=1: \quad(6 x-1)(x+1)=6 x^{2}+5 x-1 \quad \text { Error! }
\end{aligned}
$$

Trial 2: Guess a factorization of the form $\left(3 x+r_{1}\right)\left(2 x+r_{2}\right)$. The values of $r_{1}$ and $r_{2}$ are such that $r_{1} r_{2}=-1$.

$$
\begin{array}{lll}
\text { Try: } r_{1}=1, r_{2}=-1: & (3 x+1)(2 x-1)=6 x^{2}-x-1 & \text { Error! } \\
\text { Try: } r_{1}=-1, r_{2}=1: & (3 x-1)(2 x+1)=6 x^{2}+x-1 & \text { Success! }
\end{array}
$$

The factorization is,

$$
6 x^{2}+x-1=(3 x-1)(2 x+1)
$$

## Important Points

It is stated in the definition that a polynomial is reducible if it can written as a product of polynomials of smaller degree. In the factorization

$$
x^{2}+1=\frac{1}{2}\left(2 x^{2}+2\right)
$$

it is true that $\frac{1}{2}$ has smaller degree than $x^{2}+1$, but the other factor, $2 x^{2}+2$, has degree equal to that of $x^{2}+1$. Therefore, the above factorization does not mean $x^{2}+1$ is reducible-it's irreducible.

This is why the phrase "of lesser degree" was included in the definition of reducible.

No. Simply factoring a polynomial by taking out a numerical constant says nothing about whether that polynomial is reducible or not. This particular polynomial $2 x^{2}+2 x$ is reducible because

$$
2 x^{2}+2 x=2 x(x+1)
$$

We have factored a degree two polynomial into a product of two degree 1 polynomials. Each factor has degree less than the original polynomial $2 x^{2}+2$.

