THE UNIVERSITY OF AKRON Mathematics and Computer Science

Lesson 6: Dividing & Factoring Polynomials

Directory

• Table of Contents, PDPA I am DS • Begin Lesson 6 $\begin{array}{c} \mathbf{n} \ \mathbf{0} \\ \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\ a^{3}a^{4} = a^{7} \ (ab)^{10} = a^{10}b^{10} \\ -(ab - (3ab - 4)) = 2ab - 4 \end{array}$ $\sqrt{(ab)^3(a^{-1}+b^{-1})} = (ab)^2(a+b)^3$ $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ $2x^2 - 3x - 2 = (2x + 1)(x - 2)$ $\frac{1}{2}x + 13 = 0 \implies x = -26$ $G = \{ (x, y) \mid y = f(x) \}$ f(x) = mx + b $y = \sin x$ Copyright ©1995-2000 D. P. Story Last Revision Date: 2/2/2000



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6. Dividing & Factoring Polynomials

6.1. Polynomials: A Quick Review

A **polynomial** in x is an algebraic expression that can be built up through any (finite) combination of additions, subtractions, and multiplications of the symbol x with itself and with numerical constants. The **degree** of the polynomial is the value of the highest exponent.

Illustration 1. Examples of Polynomials in x.

- (a) 2x+1 has degree 1. A degree one polynomial is sometimes called **linear** because its graph is a straight line.
- (b) $5x^2 4x + 3$ has degree 2. A degree two polynomial is called a **quadratic** polynomial.

(c)
$$7x^3 - 4x^2 - \frac{2}{3}x + \frac{8}{9}$$
 has degree 3. This is called a **cubic** polynomial.

(d)
$$x^4 + x$$
 has degree 4.
(e) $x^{45} - 5x^{34} + x^3 + 1$ has degree 45.
(f) $-3y^3 + 2y - y + \frac{1}{2}$ is a polynomial in y of degree 3.

(g) As a convenience, a constant is considered to be a polynomial of degree 0. Thus, the algebraic expression 3 may be interpreted as a polynomial of degree 0.

Sometimes we symbolically denote a polynomial in x by notations such as P(x) or Q(x); polynomials in some other variable such as ywould be denoted similarly: P(y) and Q(y). If P(x) is a polynomial in x, then we write (type) deg(P(x)) to refer to the *degree* of P(x).

Polynomials arise naturally in many branches of mathematics and engineering. Polynomials are the basic building blocks used to create numerical approximations such as the ones used by your hand-held calculator. The graphs of certain polynomials have properties that man exploits to create many useful everyday conveniences such as flashlights and satellite dishes.

One of the reasons for the importance of polynomials—and quotients of polynomials—is that their values can be computed by elementary

arithmetic operations: addition, subtraction, multiplication and division. These are the operations a computer is designed to perform quickly and efficiently.

The sum and product of polynomials is again a polynomial. Illustrate this assertion by doing the next exercise. Classify each "answer" as a polynomial and state its degree.

EXERCISE 6.1. The Algebra of Polynomials. Perform the indicated operations and classify the results.

(a)
$$(4x^3 - 6x^2 + 2x + 1) + (2x^3 - 3x + 4)$$

(b) $(7x^5 - 4x^3 + 12x - 4) - (5x^5 + 3x^4 + 4x^3 + 2x - 4)$
(c) $(3x - 2)(x^2 - 2x + 1)$
(d) $(x^2 - 4)(x^2 + 4)$

Adding and multiplying polynomials were covered by the general methods described in LESSON 3 (addition) and in LESSON 5 (multiplication). In the next paragraph we look at division of polynomials.

6.2. Polynomial Division

In this paragraph, we will be primarily interested in dividing polynomials. If N(x) and D(x) are polynomials in x then the expression

$$\frac{N(x)}{D(x)}\tag{1}$$

is called a *rational expression* or a *rational function*.

Illustration 2. Examples of Rational Expressions.

(a)
$$\frac{2x+1}{5x^2-4x+2}$$
 (b) $\frac{3x^3-2x^2+7x+1}{x^2+3x-2}$
(c) $\frac{4x^{12}-7x^9+1}{x^3}$ (d) $\frac{4x^4-3x^3+2x^2+8x-3}{x+3}$

• Polynomial Division Explained

Now turning to the problem of computing (1), generally, we can only divide a ratio of polynomials when the *when the degree of the numerator is* greater than or equal to *the degree of the denominator*. In symbols

If
$$\deg(N) \ge \deg(D)$$
, then we can divide $\frac{N(x)}{D(x)}$

In this case, when we do divide, then the result looks like this

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \tag{2}$$

where Q(x) and R(x) are polynomials and deg (R(x)) < deg(D(x)). An interesting and important fact to remember is that the representation on the right-hand side of equation (2) is *unique*. This fact will be exploited in the section on factoring.

Terminology. Given the representation in equation (2), then

- N(x) is called the **dividend**;
- D(x) is called the **divisor**;

- Q(x) is called the **quotient**;
- R(x) is called the **remainder**.

This is the same terminology used in long division of numbers.

In ILLUSTRATION 2, the numerator has less degree than the denominator in example (a); consequently, we cannot divide these two polynomials. In examples (b)–(d), the degree of the numerator is equal to or greater than the degree of the denominator; in each of these examples we can further expand the expression by dividing denominator into the numerator.

Division Algorithm. How to divide one polynomial by another and obtain a result of the form (2)

- (a) Arrange the terms of N(x) and D(x) so that the powers of the terms are listed in descending order.
- (b) Divide the first term of the dividend by the first term of the divisor. This gives the first term of the quotient.

- (c) Now, multiply the term of the quotient just computed by the divisor, and subtract this product from the dividend. The result is the remainder.
 - ▷ If the degree of the remainder is less than the degree of the divisor, D(x), you are done!
 - \triangleright If the degree of the remainder is *not* less than the divisor, D(x), continue by using the remainder just obtained as a new dividend, repeat; i.e., go to (b) to compute the next term of the quotient.

The next example shows how to perform polynomial division. Study this example closely; with a paper and pencil, slowly work through the example—follow my calculations as I work through the DIVISION ALGORITHM.

EXAMPLE 6.1. Divide the expression: $\frac{2x^3 - 3x^2 + 6x - 4}{x - 1}$.

Again, work through these two examples with pencil and paper. Strive to understand how each entry is determined from the DIVISION AL-GORITHM.

EXAMPLE 6.2. Divide denominator into the numerator in parts (b) and (d) in ILLUSTRATION 2.

Part (c) of ILLUSTRATION 2 can be handled differently than parts (b) and (d). Why? Read the example to find out why.

EXAMPLE 6.3. Divide $\frac{4x^{12} - 7x^9 + 1}{x^3}$ that appeared in part (c) of ILLUSTRATION 2.

Having done a few examples of polynomials, its time for you to try a few. Follow the DIVISION ALGORITHM and my examples.

EXERCISE 6.2. Divide:
$$\frac{2x^3 - 3x^2 + x - 1}{x - 2}$$
.
EXERCISE 6.3. Divide: $\frac{4x^4 + 2x^2 + x + 1}{x^2 + 1}$.

In the next exercise the remainder is zero. Do you recall what that means?

EXERCISE 6.4. Divide:
$$\frac{x^3+1}{x+1}$$
.

6.3. Factoring Polynomials: Motivation

Factoring is the reverse process to expanding:

$$(x+1)(x+2) = x^2 + 3x + 2.$$
 (3)

Reading the equation from left-to-right, we are *expanding* the polynomial (x + 1)(x + 2); reading the equation from right-to-left we are *factoring* the polynomial $x^2 + 3x + 2$.

Illustration 3. Factoring has a variety of uses in mathematics. Here is a set of examples to illustrate that assertion.

(a) *Simplification* of algebraic expressions:

$$\frac{x^3 + 3x^2 + 2x}{x+1} = \frac{x(x^2 + 3x + 2)}{x+1} \quad \triangleleft \text{ factor a common } x$$
$$= \frac{x(x+1)(x+2)}{x+1} \quad \triangleleft \text{ factor again by (3)}$$
$$= x(x+2) \quad \triangleleft \text{ nicely simplified!}$$

Surely you agree, the last expression is preferable to the first.
(b) Solving Equatons: For what values of x is x²+3x+2 equal zero? That is, solve the equation

$$x^2 + 3x + 2 = 0$$

Since $x^2 + 3x + 2 = (x + 1)(x + 2)$ it is now obvious that the only values of x that make $x^2 + 3x + 2$ equal zero are x = -1 and x = -2.

(c) *Simplification* for the purpose of numerical calculations. Suppose you wanted to compute, on your calculator, values of the polynomial

 $x^4 + 4x^3 + 6x^2 + 4x^{+1}$

for x = 3.23344, -27.3234, and 34.000123. Even with the aid of your calculator this would be a tedious task, and there is a very good chance that you will make errors. Suppose I told you that the above expression is nothing more than

 $(x+1)^4$

Which one would you use to make the calculations?

Summary. Factorization tends to simplify, reduce the number of arithmetical operations performed, and often times yields valuable information about the behavior of the expression.

6.4. Factoring Polynomials: Theory

In this section we discuss some important background information and theory: Reducible versus Irreducible Polynomials; the FUNDAMENTAL THEOREM OF ALGEBRA; and Roots and Factors Related.

These are important and fundamental concepts you should try to understand.

• Reducible versus Irreducible Polynomials

Some polynomials can be factored, others cannot. Naturally we have a terminology for each situation.

A polynomial that can be factored into a product of polynomials of smaller degree is called a **reducible**; otherwise, the polynomial is called **irreducible**.

Illustration 4. Reducible and Irreducible Polynomials.

(a) The polynomial x^2+3x+2 is reducible because it can be factored into factors of smaller degree:

$$x^{2} + 3x + 2 = (x+1)(x+2).$$

We have factored a second degree polynomial into a product of linear polynomials.

(b) The polynomial $x^3 - 3x^2 + 3x - 1$ is reducible because

$$x^3 - 3x^2 + 3x - 1 = (x - 1)^3.$$

Here we say that (x - 1) is a linear factor of multiplicity 3. (c) The polynomial $x^3 + x^2 + x + 1$ is reducible because

$$x^{3} + x^{2} + x + 1 = (x + 1)(x^{2} + 1).$$

This polynomial factors into linear and quadratic factors.
(d) The polynomials x² + 1 and x² - x + 1 are irreducible; that is, it cannot be factored any further. How do I know that? Read on!

Question. It was stated in the last illustration that x^2+1 is irreducible. Suppose I factor it as follows: $x^2 + 1 = \frac{1}{2}(2x^2 + 2)$. Since I have **factored** it into a product of polynomials (the constant $\frac{1}{2}$ may be considered a polynomial of degree 0), does this mean that $x^2 + 1$ is

reducible? Review the definitions of reducible and irreducible before you respond.

Question. Consider the factorization: $2x^2 + 2x = (2)(x^2 + x)$. We have factored a degree two polynomial into a product of a degree 0 polynomial and a degree 2 polynomial. In light of the discussion in the previous Question, does this mean that $2x^2 + 2x$ is irreducible? (a) Yes (b) No

• The Fundamental Theorem of Algebra

According to the FUNDAMENTAL THEOREM OF ALGEBRA, any polynomial of degree greater than zero can be factored into a product of linear and irreducible quadratic factors.

Illustration 5. For example,

$$x^{4} + 2x^{3} + 2x^{2} + 2x + 1 = (x+1)^{2}(x^{2}+1).$$
(4)

Linear Factors: The factor $(x + 1)^2$ is called a *linear factor* because x + 1, the base of the power, is a degree 1 polynomial (degree 1 = linear). The presence of $(x + 1)^2$ in the factorization, of course, means that (x + 1) appears twice in the factorization

$$x^{4} + 2x^{3} + 2x^{2} + 2x + 1 = (x+1)(x+1)(x^{2}+1).$$

but hardly ever write it this way. Here, we say that (x + 1) is a linear factor of *multiplicity* 2, meaning it appears twice in the factorization.

Irreducible Quatdratic Factors: The other factor, $(x^2 + 1)$, in equation (4) is a second degree polynomial, or a quadratic polynomial. This particular one is irreducible. The exponent of the factor $(x^2 + 1) = (x^2 + 1)^1$ is 1 meaning this factor has multiplicity 1; there is only 1 factor of this type present in the factorization (4).

Thus, the factorization of the polynomial $x^4 + 2x^3 + 2x^2 + 2x + 1$ given in equation (4) is the one described in the FUNDAMENTAL THEOREM OF ALGEBRA; that is,

$$x^{4} + 2x^{3} + 2x^{2} + 2x + 1 = (x+1)^{2}(x^{2}+1).$$

is a factorization into linear and irreducible quadratics factors. EXERCISE 6.5. Consider the factorization:

$$x^{3}(2x+1)^{4}(x^{2}+1)^{5}(5-2x)^{3}(x^{2}+x+1).$$

Classify each factor as linear or irreducible quadratic and state the multiplicity of each factor.

• Roots and Linear Factors Related

There is a relationship between a root of a polynomial and its linear factors. Let's begin by recalling what a root of a polynomial is. Let

$$P(x) = ax^3 + bx^2 + cx + d$$

be any polynomial. (For illustrative purposes, I've just written a third degree polynomial.) A **root** of P(x) is any number r such that

$$P(r) = ar^3 + br^2 + cr + d = 0$$
(5)

that is, a **root** is any number r that causes the polynomial to evaluate to zero. Another way of thinking of a root is, a root is any solution to the equation:

$$ax^3 + bx^2 + cx + d = 0.$$

Illustration 6. The numbers -1 and -2 are roots of the polynomial $P(x) = x^2 + 3x + 2$ since

▷
$$P(-1) = (-1)^2 + 3(-1) + 2 = 1 - 3 + 2 = 0$$

▷ $P(-2) = (-2)^2 + 3(-2) + 2 = 4 - 6 + 2 = 0$

In the next lesson, LESSON 7, we discuss techniques for finding roots of polynomials. There, we will discuss methods of solving equations.

What is the relation between roots and linear factors?

 \triangleright (x-r) a factor implies r is a root. Let P(x) be any polynomial and r a number. Clearly, if (x-r) is a *factor* or P(x), then r is a root of P(x). Indeed, if we assume (x-r) is a factor of P(x) then

$$P(x) = Q(x)(x-r)$$

for some polynomial Q(x); that is, (x-r) appears in the factorization of P(x). Now you can see that

$$P(r) = Q(r)(r-r) = 0.$$

This means that r is a root of the polynomial P(x).

 \triangleright r a root implies (x-r) a factor. Divide the polynomial P(x) by the polynomial (x-r). The result, by equation (2) will have the form:

$$\frac{P(x)}{(x-r)} = Q(x) + \frac{R(x)}{(x-r)}$$
(6)

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where the remainder R(x) has degree less than x - r. This means, in this case, that R(x) has degree zero since x - r had degree one. Degree zero polynomials are constants; thus, R(x) is, in fact, a constant. Ill call this constant R. We have then

$$\frac{P(x)}{(x-r)} = Q(x) + \frac{R}{(x-r)}.$$

Multiply both sides of the equation by x - r to obtain

$$P(x) = Q(x)(x - r) + R, \quad R = \text{constant.}$$
(7)

There is a simple interpretation for the value of R. If we replace x by r in equation (7), we see

$$P(r) = Q(r)(r-r) + R$$

or,

$$R = P(r) \tag{9}$$

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Now, if r is a root of the polynomial P(x), then from (5), P(r) = 0. But by (9), the remainder R = P(r) = 0. Substituting this into equation (7) we get,

$$P(x) = Q(x)(x - r) + 0$$
, or, $P(x) = Q(x)(x - r)$

This means that (x - r) is a factor of P(x).

We now summarize the observations of the past few paragraphs.

Summary. Roots and Linear Factors Related: Let P(x) is a polynomial and r a number. Then r is a **root** of P(x) if and only if (x - r) is a **factor** of P(x).

I still haven't explained how I know that $x^2 + 1$ is irreducible. It is clear that for any number $x, x^2 + 1 \neq 0$; therefore, the polynomial $x^2 + 1$ has **no roots**, hence, it has no **linear factors**. Now $x^2 + 1$ has degree two, if it is to be factored, it must factor into linear factors. But $x^2 + 1$ has no linear factors. Thus, $x^2 + 1$ is irreducible.

EXERCISE 6.6. Read carefully the reasoning in the previous paragraph, and apply it to the polynomial $x^4 + 1$. It is clear that for all x, $x^4 + 1 \neq 0$ and so $x^4 + 1$ has no roots, hence, has no linear factors. Can we deduce that $x^4 + 1$ is irreducible in the same way as we deduced that $x^2 + 1$ was irreducible?

6.5. Factoring Polynomials: Methods

• Factoring $x^2 + bx + c$

In this section we discuss methods of factoring a second degree polynomial with integer coefficients with leading coefficient of one:

$$x^2 + bx + c \qquad b, \ c \in \mathbb{Z}$$

If the polynomial $x^2 + bx + c$ is not irreducible, then it can be factored into a product of linear polynomials of the form:

$$x^{2} + bx + c = (x + r_{1})(x + r_{2}).$$
(10)

The problem is to determine the values of r_1 and r_2 . This can be done in one of two ways: (1) by trial and error; or (2) the QUADRATIC FORMULA. The latter method will be taken up in LESSON 7.

Let's play for a moment. Expand the right-hand side of (10):

$$x^{2} + bx + c = (x + r_{1})(x + r_{2})$$

= $x^{2} + (r_{1} + r_{2})x + r_{1}r_{2}$.

Thus, if $x^2 + bx + c$ can be factored into $(x + r_1)(x + r_2)$, then

 $r_1 + r_2 = b$ $r_1 r_2 = c$

Let's now summarize the results so far in the form of a shadow box.

How to factor $x^2 + bx + c$, where $b, c \in \mathbb{Z}$ We try to find two number r_1 and r_2 such that $r_1 + r_2 = b$ $r_1r_2 = c$ In this case, $x^2 + bx + c = (x + r_1)(x + r_2)$.

Strategy. Given $x^2 + bx + c$, where b and c are integers: First list all pairs of **integers** r_1 and r_2 the product of which is c. Then among all those pairs of numbers, r_1 and r_2 , thus listed, choose the pair whose sum is b—choose that pair r_1 and r_2 such that $r_1 + r_2 = b$.

Let's go to the examples.

EXAMPLE 6.4. Factor each of the following polynomials. (a) $x^2 - x - 2$ (b) $x^2 + 5x + 6$ (c) $4 - 3x - x^2$

Solve the next exercise using good notation. Be neat, be organized. Take *pride in your work*!

EXERCISE 6.7. Factor each of the following using the recommended strategy as illustrated in EXAMPLE 6.4.

(a)
$$x^2 + 7x + 10$$
 (b) $x^2 - 7x + 10$ (c) $x^2 - 3x - 10$
(d) $x^2 + 3x - 10$ (e) $x^2 + 11x + 10$ (f) $x^2 - 9x - 10$

Here's a few more. Don't forget the strategy and the standard methods as illustrated in EXAMPLE 6.4.

EXERCISE 6.8. Factor each of the following. (a) $x^2 + 4x - 12$ (b) $x^2 + 3x - 18$ (c) $x^2 - 10x + 21$ (d) $x^2 + 7x - 8$ (e) $x^2 - 2x + 1$ (f) $2x^2 + 8x + 8$

• Factoring $x^2 - a^2$

Polynomials that are in the form of a difference of two squares are easy to factor. From equation (3) of LESSON 5 we have

$$x^{2} - a^{2} = (x - a)(x + a)$$
(11)

The problem students have is recognizing the presence of such a polynomial. It's a matter of training you eyes and brain to work together.

Illustration 7. Difference of Two Squares.

EXERCISE 6.9. (Skill Level 0) Factor each of the following. (a) $x^2 - 9$ (b) $x^2 - 12$ (c) $x^2 - 17$ (d) $25 - x^2$

Factorization formula (11) came from expansion formula (3) of LES-SON 5. The expansion formula is actually more general: We can factor *any* difference of two squares—as illustrated below.

Illustration 8. Difference of two squares.

EXERCISE 6.10. Factor each of the following using (11). (a) $4x^2 - 9$ (b) $5x^2 - 3$ (c) $x^4 - 25$

The factorization formula (11) can be applied whenever we have a difference of two squares.

EXERCISE 6.11. Factor each of the following differences of squares and simplify when possible. Note: 'simplify' \neq 'expand'.

(a)
$$x^2 - (x-1)^2$$
 (b) $x^2y^2 - 4$ (c) $\frac{(2x-1)^2 - (2x+1)^2}{x}$

• Factoring
$$ax^2 + bx + c$$

We now turn to the problem of factoring a quadratic polynomial of the form

$$ax^2 + bx + c \qquad a, b, c \in \mathbb{Z}$$
(12)

The trial and error methods illustrated earlier can be utilized to factor (12). Generally, if the polynomial is *not* irreducible, its factorization of would look like this:

$$ax^{2} + bx + c = (q_{1}x + r_{1})(q_{2}x + r_{2})$$
(13)

If we multiply out the right-hand side of (13) that will give us clues how to find the numbers q_1 , q_2 , r_1 , and r_2 :

$$ax^{2} + bx + c = (q_{1}x + r_{1})(q_{2}x + r_{2})$$
$$= q_{1}q_{2}x^{2} + (q_{1}r_{2} + r_{1}q_{2})x + r_{1}r_{2}$$

Thus, we seek numbers such that

$$q_1q_2 = a$$
 $r_1r_2 = c$ $q_1r_2 + q_2r_1 = b$

The task of finding the numbers q_1 , q_2 , r_1 , and r_2 may be easy or difficult.

Important Point. It is known that if the polynomial $ax^2 + bx + c$ has only *rational roots*—roots that are rational numbers—then q_1, q_2, r_1 , and r_2 are integers. However, a quadratic polynomial such as (12) may have irrational roots.

In this section, we look at quadratics having rational roots, hence the factors we seek have integer coefficients. The other case can be handled by use of the binomial formula taken up in LESSON 7.

Strategy. To factor $ax^2 + bx + c$ into to linear factors

$$ax^{2} + bx + c = (q_{1}x + r_{1})(q_{2} + r_{2})$$

(a) List all *integer pairs* q_1 and q_2 such that $q_1q_2 = a$, the coefficient of the x^2 term.

- (b) List all *integer pairs* r_1 and r_2 such that $r_1r_2 = c$, the constant term.
- (c) Find that combination of q_1 , q_2 and r_1 , r_2 such that $q_1r_2+q_2r_1 = b$, the coefficient of the cross-product term.

This can usually be carried out by first choosing candidates for q_1 and q_2 then trying all combinations for r_1 and r_2 . If that fails, try another choice for q_1 and q_2 . (I told you it was 'trial and error.')

Here's an example of the scheme just described.

EXAMPLE 6.5. Factor
$$6x^2 + x - 1$$
.

In the exercises below, use EXAMPLE 6.5 as a guide to factoring.

EXERCISE 6.12. Factor $8x^2 + 2x - 1$.

EXERCISE 6.13. Factor $6x^2 - 5x - 6$.

In LESSON 7 we'll review techniques of solving equations. Solving polynomial equations has applications to factorization. (Recall the paragraph on Roots and Linear Factors Related.) At that time we will

be able to factor quadratics with irrational roots as well as rational roots.

• Factoring $x^3 \pm a^3$

The sum or difference of cubes is easy to factor so we'll finish this lesson by factoring this type. The formula we shall use is

$$b^{3} - a^{3} = (b - a)(b^{2} + ab + a^{2})$$

$$b^{3} + a^{3} = (b + a)(b^{2} - ab + a^{2})$$
(14)

The major problem of students is recognizing the presence of cubes.

Illustration 9. Factor each of the following.

(a)
$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$
. Here $a = 2$ in (14).
(b) $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$.
(c) $8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$. Here $b = 2x$ and $a = 1$.
(d) $x^3y^3 + 1 = (xy + 1)(x^2y^2 - xy + 1)$. Here $b = xy$ and $a = 1$.

EXERCISE 6.14. (Skill Level 0) Factor each of the following. (a) $x^3 - 1$ (b) $27x^3 - 8$ (c) $8y^6 + 27$

EXERCISE 6.15. (Skill Level 0.5) Factor and simplify each. (a) $\frac{x^3 - 1}{x - 1}$ (b) $\frac{(8x^3 - 1)(x^2 + 3x + 2)}{x^2 - 1}$ (c) $\frac{8x^3 - 27}{4x^2 - 9}$

This is the end of LESSON 6 on dividing polynomials and factoring. To continue, go to LESSON 7.

Solutions to Exercises

6.1. Solutions:

(a) Combine $(4x^3 - 6x^2 + 2x + 1) + (2x^3 - 3x + 4)$.

$$(4x^3 - 6x^2 + 2x + 1) + (2x^3 - 3x + 4) = 6x^3 - 6x^2 - x + 5.$$

This is a polynomial of degree 3.

(b)
$$(7x^5 - 4x^3 + 12x - 4) - (5x^5 + 3x^4 + 4x^3 + 2x - 4)$$

 $(7x^5 - 4x^3 + 12x - 4) - (5x^5 + 3x^4 + 4x^3 + 2x - 4)$
 $= 2x^5 - 3x^4 - 8x^3 + 10x$

which is a polynomial of degree 5.

(c) $(3x-2)(x^2-2x+1)$. Use the General Multiplication Rule to obtain

$$(3x-2)(x^2-2x+1)$$

= $(3x)(x^2) - (3x)(2x) + (3x) - 2(x^2) + 2(2x) - 2$
= $3x^3 - 6x^2 + 3x - 2x^2 + 4x - 2$
= $3x^3 - 8x^2 + 7x - 2$

Alternately, you can expand as follows:

$$\frac{x^{2} - 2x + 1}{3x - 2} \\
\frac{3x^{3} - 6x^{2} + 3x}{-2x^{2} + 4x - 2} \\
\frac{3x^{3} - 8x^{2} + 7x - 2}{3x^{3} - 8x^{2} + 7x - 2} \iff \text{The is a polynomial of degree 3.}$$

(d) $(x^2 - 4)(x^2 + 4) = x^4 - 16$ by (3) in LESSON 5. A polynomial of degree 4.

Exercise 6.1.

6.2. Solution: Just follow the DIVISION ALGORITHM!

lotient

vide $2x^3$ by x to get $2x^2$

ultiply $2x^2$ by x-2 and place it here

btract; (b) divide x^2 by x to get x

$$\triangleleft$$
 (c) multiply x by $x - 2$ and place it here

btract; (b) divide 3x by x to get 3

ultiply 3 by x - 2 and place it here

btract; done; this is the remainder

Interpretation of Calculations: x - 2 divides into $2x^3 - 3x^2 + x - 1$ $\dots 2x^2 + x + 3$ times with a remainder of 5. Thus,

$$\frac{2x^3 - 3x^2 + x - 1}{x - 2} = 2x^2 + x + 3 + \frac{5}{x - 2}$$

Compare this equation with (2).

Exercise 6.2.

6.3. Solution: Just follow the DIVISION ALGORITHM!

\triangleleft the quotient

 $\begin{array}{rcl} 4+2x^2+x+1 & < (b) \text{ divide } 4x^4 \text{ by } x^2 \text{ to get } 4x^2 \\ \hline 4+4x^2 & < (c) \text{ multiply } 4x^2 \text{ by } x^2+1 \text{ and place it here} \\ \hline -2x^2+x+1 & < (c) \text{ subtract; } (b) \text{ divide } -2x^2 \text{ by } x^2 \text{ to get } -2 \\ \hline -2x^2 & -2 & < (c) \text{ multiply } -2 \text{ by } x^2+1 \text{ and place it here} \\ \hline 4 & (c) \text{ subtract; done; this is the remainder} \end{array}$

Interpretation of Calculations: $x^2 + 1$ divides into $4x^4 + 2x^2 + x + 1$... $4x^2 - 2$ times with a remainder of x + 3. Thus,

$$\frac{4x^4 + 2x^2 + x + 1}{x^2 + 1} = 4x^2 - 2 + \frac{x + 3}{x^2 + 1}$$

Compare this equation with (2).

Exercise 6.3.

6.4. Solution:



- $-x^2 x$ \triangleleft (c) multiply -x by x + 1 and place it here
 - x+1 < (c) subtract; (b) divide x by x to get 1
 - x+1
 (c) multiply 1 by x + 1 and place it here
 - \triangleleft (c) subtract; done; this is the remainder

Interpretation of Calculations: x + 1 divides into $x^3 + 1 \dots x^2 - x + 1$ times with a remainder of 0. Thus,

$$\frac{x^3 + 1}{x + 1} = x^2 - x + 1 + \frac{0}{x + 1} = x^2 - x + 1$$

This means that x + 1 evenly divides into $x^3 + 1$. If we multiply both sides of the equations by x + 1 we get

$$x^3 + 1 = (x+1)(x^2 - x + 1)$$

The division process can be used as a technique to factor a polynomial. Exercise 6.4. \bullet

6.5. *Answers*:

Factor	Classification	Multiplicity
x^3	linear	3
$(2x+1)^4$	linear	4
$(x^2+1)^5$	irreducible quadratic	5
$(5-2x)^3$	linear	3
$(x^2 + x + 1)$	irreducible quadratic	1

Exercise 6.5. \blacksquare

6.6. No! $x^4 + 1$ has no linear factors, but, by the FUNDAMENTAL THEOREM OF ALGEBRA it can be factored into a product of linear factors and irreducible quadratic factors. As it doesn't have any linear factors, and its degree is 4, it is must be possible to factor it into a product of two irreducible quadratic factors.

Can you find these two factors?

Exercise 6.6. \blacksquare

6.7. Answers: Hopefully, you used the method illustrated in EXAM-PLE 6.4.

(a)
$$x^2 + 7x + 10 = (x + 2)(x + 5).$$

(b) $x^2 - 7x + 10 = (x - 2)(x - 5).$
(c) $x^2 - 3x - 10 = (x + 2)(x - 5).$
(d) $x^2 + 3x - 10 = (x - 2)(x + 5).$
(e) $x^2 + 11x + 10 = (x + 1)(x + 10).$
(f) $x^2 - 9x - 10 = (x + 1)(x - 10).$

Did I list all possible combinations, or did I miss one or two?

Exercise 6.7.

6.8. Answers:
(a)
$$x^2 + 4x - 12 = (x - 2)(x + 6)$$
.
(b) $x^2 + 3x - 18 = x + 6(x - 3)$.
(c) $x^2 - 10x + 21 = (x - 3)(x - 7)$.
(d) $x^2 + 7x - 8 = (x - 1)(x + 8)$.
(e) $x^2 - 2x + 1 = (x - 1)(x - 1) = (x - 1)^2$ a perfect square!
(f) $2x^2 + 8x + 8 = 2(x^2 + 4x + 4) = 2(x + 2)(x + 2) = 2(x + 2)^2$ again
a perfect \Box !

Exercise 6.8. \blacksquare

6.9. Answers: (a) x² - 9 = (x - 3)(x + 3). (b) x² - 12 = (x - √12)(x + √12) = (x - 2√3)(x + 2√3). You did simplify, didn't you? (c) x² - 17 = (x - √17)(x + √17). (d) 25 - x² = (5 - x)(5 + x) = -(x - 5)(x + 5). We usually like the x-term to come first.

These factorizations are rather boring. Exercise 6.9.

6.10. *Answers*:

(a)
$$4x^2 - 9 = (2x - 3)(2x + 3).$$

(b) $5x^2 - 3 = (\sqrt{5}x - \sqrt{3})(\sqrt{5}x + \sqrt{3}).$
(c) $x^4 - 25 = (x^2 - 5)(x^2 + 5) = (x - \sqrt{5})(x + \sqrt{5})(x^2 + 5).$
Exercise 6.10.

6.11. Solutions:

(a)
$$x^{2} - (x - 1)^{2} = [x - (x - 1)][x + (x - 1)] = [1][2x - 1] = 2x - 1$$

(b) $x^{2}y^{2} - 4 = (xy - 2)(xy + 2)$
(c) Factor: $\frac{(2x - 1)^{2} - (2x + 1)^{2}}{x}$.
 $\frac{(2x - 1)^{2} - (2x + 1)^{2}}{x}$
 $= \frac{[(2x - 1) - (2x + 1)][(2x - 1) + (2x + 1)]}{x}$
 $= \frac{[-2][4x]}{x} = -\frac{8x}{x} < \text{cancel the } x!$
 $= [-8]$

That simplified down nicely!

Exercise 6.11.

6.12. *Solution*:

$$8x^2 + 2x - 1 = (4x - 1)(2x + 1).$$

Note: You really don't need the answers, just multiply out your factorization. If it expands to $8x^2 + 2x - 1$, you are right!

Exercise 6.12.

6.13. Solution:

$$6x^2 - 5x - 6 = (3x + 2)(2x - 3)$$

Exercise 6.13. \blacksquare

6.14. *Answers*:

(a)
$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

(b) $27x^3 - 8 = (3x - 2)(9x^2 + 6x + 4)$. $(b = 3x \text{ and } a = 2)$
(c) $8y^6 + 27 = (2y^2 + 3)(4y^4 - 6y^2 + 9)$ $(b = 2y^2 \text{ and } a = 3)$
Exercise 6.14.

6.15. Answers: (a) $\frac{x^3 - 1}{x - 1} = \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \boxed{x^2 + x + 1}$ (b) Factor and simplify: $\frac{(8x^3-1)(x^2+3x+2)}{r^2-1}$. $\frac{(8x^3-1)(x^2+3x+2)}{x^2-1}$ $=\frac{(2x-1)(4x^2+2x+1)(x+1)(x+2)}{(x-1)(x+1)}$ $= \left| \frac{(2x-1)(4x^2+2x+1)(x+2)}{(x-1)} \right|$ a > (a > a). 9 •

(c)
$$\frac{8x^3 - 27}{4x^2 - 9} = \frac{2x - 3(4x^2 + 6x + 9)}{(2x - 3)(2x + 3)} = \left\lfloor \frac{4x^2 + 6x + 9}{2x + 3} \right\rfloor$$

Exercise 6.15.

Solutions to Examples

6.1. Solution: The terms of the divisor, x - 1, and the dividend, $2x^3 - 3x^2 + 6x - 1$, are already arranged in decreasing order. (Step (a) of the Division Algorithm.

The calculations outlined in the **Division Algorithm** can be arranged into a convenient table format. The method of division and the table is similar to long division of numbers.

$$\begin{array}{r}
 2x^{2} - x + 5 \\
 x - 1 \overline{\smash{\big)}2x^{3} - 3x^{2} + 6x - 4} \\
 \underline{2x^{3} - 2x^{2}} \\
 - x^{2} + 6x - 4 \\
 \underline{-x^{2} + x} \\
 \underline{5x - 4} \\
 \underline{5x - 5} \\
 1
\end{array}$$

\triangleleft the quotient

- \triangleleft (b) divide $2x^3$ by x to get $2x^2$
- \triangleleft (c) multiply $2x^2$ by x 1 and place it here
- \triangleleft (c) subtract; (b) divide $-x^2$ by x to get -x
- \triangleleft (c) multiply -x by x 1 and place it here
- \triangleleft (c) subtract; (b) divide 5x by x to get 5
- \triangleleft (c) multiply 5 by x 1 and place it here
- \triangleleft (c) subtract; Done. This is the remainder

Interpretation of Calculations: x - 1 goes into $2x^3 - 3x^2 + 6x - 4 \dots 2x^2 - x + 5$ times with a remainder of 1. Thus,

$$\frac{2x^3 - 3x^2 + 6x - 4}{x - 1} = 2x^2 - x + 5 + \frac{1}{x - 1}$$

Compare this equation with (2). The quotient is $Q(x) = 2x^2 - x + 5$ and the remainder is R(x) = 1. Note that $\deg(R(x)) = 0 < 1$, as advertised above.

The validity of the above equation can be verified by getting a common denominator of the right-hand side of the equation and observing that it results in the left-hand side! Example 6.1.

6.2. Problem (b) Divide:
$$\frac{3x^3 - 2x^2 + 7x + 1}{x^2 + 3x - 2}$$

Solution to (b)

$$\begin{array}{r}
3x & -11 \\
x^2 + 3x - 2 \overline{\smash{\big)}3x^3 - 2x^2 + 7x + 1} \\
\underline{3x^3 + 9x^2 - 6x} \\
-11x^2 + 13x + 1 \\
\underline{-11x^2 - 33x + 22} \\
46x - 21
\end{array}$$

 \triangleleft the quotient

- \triangleleft (b) divide $3x^3$ by x^2 to get 3x
- \triangleleft (c) multiply 3x by $x^2 + 3x 2$
- \triangleleft (c) subtract; then repeat (b)
- \triangleleft (c) multiply -11 by $x^2 + 3x 2$

 \triangleleft (c) subtract; Done. The remainder

Thus,

$$\frac{3x^3 - 2x^2 + 7x + 1}{x^2 + 3x - 2} = 3x - 11 + \frac{46x - 21}{x^2 + 3x - 2}$$

Notice that the degree of the remainder is less than the degree of the divisor.

Solution to (d) Divide:
$$\frac{4x^4 - 3x^3 + 2x^2 + 8x - 3}{x + 3}$$

$$x + 3 \boxed{4x^3 - 15x^2 + 47x - 133}_{x + 3} \quad 4 \text{ the quotient} \\ 4x^4 - 3x^3 + 2x^2 + 8x - 3 \\ 4 \text{ (b) divide} \\ 4x^4 + 12x^3 \quad 4 \text{ (c) multiply} \\ \hline -15x^3 + 2x^2 + 8x - 3 \\ -15x^3 - 45x^2 \quad 4 \text{ (c) multiply} \\ \hline 47x^2 + 8x - 3 \\ 4 \text{ (c) multiply} \\ \hline 47x^2 + 8x - 3 \\ 4 \text{ (c) multiply} \\ \hline -133x - 3 \\ 4 \text{ (c) multiply} \\ \hline -133x - 3 \\ 4 \text{ (c) multiply} \\ \hline 396 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 3 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 399 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 390 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 390 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 390 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 390 \\ 4 \text{ (c) multiply} \\ \hline 40 \\ -133x - 30 \\ -130 \\$$

Thus,

$$\frac{4x^4 - 3x^3 + 2x^2 + 8x - 3}{x + 3} = 4x^3 - 15x^2 + 47x - 133 + \frac{396}{x + 3}$$

Example 6.2. \blacksquare

6.3. Solution: Here the denominator, the divisor, is a monomial: a single term. In this case, we don't have to carry out the elaborate division process. We simply ...

$$\frac{4x^{12} - 7x^9 + 1}{x^3} = \frac{4x^{12}}{x^3} - \frac{7x^9}{x^3} + \frac{1}{x^3}$$
$$= 4x^9 - 7x^6 + \frac{1}{x^3}.$$

That is, we divide by separation of fractions. Thus,

$$\frac{4x^{12} - 7x^9 + 1}{x^3} = 4x^9 - 7x^6 + \frac{1}{x^3}$$

When a shorter method works ... use the shorter method!

Example 6.3. \blacksquare

6.4. Solutions:

(a) Factor $x^2 - x - 2$. Here we seek two numbers r_1 and r_2 such that $r_1 + r_2 = -1$ and $r_1 r_2 = -1$.

Solution: List all pairs of integers the product of which is -2.

Therefore, the factorization is for $r_1 = -2$ and $r_2 = 1$:

$$x^{2} - x - 2 = (x + r_{1})(x + r_{2})$$

= $(x + (-2))(x + 1)$
= $(x - 2)(x + 1)$.

Or,

$$x^{2} - x - 2 = (x - 2)(x + 1).$$

(b) Factor $x^2 + 5x + 6$.

Solution: We follow the strategy suggested above. List all pairs of integers the product of which is 6, and we search for that pair whose sum is 5.

Notice that I did not list out all possible combinations. I stopped as soon as I found the proper pair: $r_1 = 2$ and $r_2 = 3$. The factorization is

$$x^{2} + 5x + 6 = (x + 2)(x + 3).$$

(c) Factor
$$4 - 3x - x^2$$

Solution: This has a slight twist. The coefficient of the x^2 term is not one. We just factor out -1 to get

$$4 - 3x - x^2 = -(x^2 + 3x - 4)$$

and factor $x^2 + 3x - 4$. Again, we follow the strategy suggested above. List all pairs of integers the product of which is -4, and we search for that pair whose sum is 3.

The factorization is

$$4 - 3x - x^{2} = -(x^{2} + 3x - 4) = -(x - 1)(x + 4).$$

Or, we could, perhaps write is as follows:

$$4 - 3x - x^{2} = (1 - x)(x + 4).$$

Example 6.4.

6.5. Solution: This is not a pretty method. The problem is to factor $6x^2 + x - 1$.

Trial 1: Guess a factorization of the form $(6x+r_1)(x+r_1)$. The values of r_1 and r_2 are such that $r_1r_2 = -1$.

Try: $r_1 = 1, r_2 = -1$: $(6x + 1)(x - 1) = 6x^2 - 5x - 1$ Error! Try: $r_1 = -1, r_2 = 1$: $(6x - 1)(x + 1) = 6x^2 + 5x - 1$ Error!

Trial 2: Guess a factorization of the form $(3x + r_1)(2x + r_2)$. The values of r_1 and r_2 are such that $r_1r_2 = -1$.

Try:
$$r_1 = 1, r_2 = -1$$
: $(3x + 1)(2x - 1) = 6x^2 - x - 1$ Error!
Try: $r_1 = -1, r_2 = 1$: $(3x - 1)(2x + 1) = 6x^2 + x - 1$ Success!

The factorization is,

$$6x^2 + x - 1 = (3x - 1)(2x + 1).$$

Example 6.5.

Important Points

Important Points (continued)

It is stated in the definition that a polynomial is reducible if it can written as a product of polynomials of *smaller degree*. In the factorization

$$x^2 + 1 = \frac{1}{2}(2x^2 + 2),$$

it is true that $\frac{1}{2}$ has smaller degree than $x^2 + 1$, but the other factor, $2x^2 + 2$, has degree *equal* to that of $x^2 + 1$. Therefore, the above factorization does not mean $x^2 + 1$ is reducible—it's irreducible.

This is why the phrase "of lesser degree" was included in the definition of reducible. Important Point

Important Points (continued)

No. Simply factoring a polynomial by taking out a numerical constant says nothing about whether that polynomial is reducible or not. This particular polynomial $2x^2 + 2x$ is reducible because

$$2x^2 + 2x = 2x(x+1).$$

We have factored a degree two polynomial into a product of two degree 1 polynomials. Each factor has degree *less than* the original polynomial $2x^2 + 2$. Important Point •